On certain integral operators (New Extension of Historical Theorems for Univalent Function Theory)

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Abstract. Let A be the class of functions $f(z)$ which are analytic in the open unit disk U with $f(0)=0$ and $f'(0)=1$. The object of the present paper is to consider the subordinations of certain integral operators for functions belonging to the class A.

1 Introduction

Let A be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. Let S^* be the subclass of A consisting of functions $f(z)$ which are satisfy

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0 \qquad (z\in U).$$

A function $f(z)$ in S^* is said to be starlike in U. For functions $f(z)$ and $g(z)$ belonging to A, we say that $f(z)$ is subordinate to $g(z)$ if there exists the function $w(z)$ which is analytic in U with $w(0)=0$ and $|w(z)|<1(z\in U)$ such that $f(z)=g(w(z))$. We denote this subordination by $f(z)\prec g(z)$. By virtue of the definition for subordinations, we know that: (i) The subordination $f(z)\prec g(z)$ implies that $f(0)=g(0)$ and $f(U)\subset g(U)$. (ii) If $g(z)$ is univalent in U, then the subordination $f(z)\prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$.
For a function $f(z)\in A$, we consider the integral operator $I(f(z))$ given by

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$$I_{\alpha.\beta}(f(z)) = \left\{\frac{\alpha+\beta}{z^\beta}\int_0^z t^{\beta-1}f(t)^\alpha dt\right\}^{1/\alpha}, \tag{1.2}$$

where $\alpha \in \mathbb{C}$,$\alpha \neq 0$, and $\beta \in \mathbb{C}$.

Remark 1. (i) Libera [2] showed that if $f(z) \in S^*$, then $I_{1,1}(f(z)) \in S^*$.
(ii) Bernardi [1] showed that if $f(z) \in S^*$, then $I_{1,\beta}(f(z)) \in S^*$when $\beta = 1, 2, 3, \ldots$.
(iii) Miller, Mocanu and Reade [6] showed that if $f(z) \in S^*$, then $I_{\alpha,\beta}(f(z)) \in S^*$when $\alpha > 0$ and $\beta \geq 0$.

To consider our integral operators for $f(z) \in A$, we have to recall here the following lemmas.

Lemma 1. ([3]) *Let $f(z)$ and $g(z)$ belong to A and $g(z)$ be univalent in $\overline{U} = U \cup \partial U$. If there exists points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that*

$$f(|z| < |z_0|) \subset g(U) \quad \textit{and} \quad f(z_0) = g(\zeta_0),$$

then $z_0 f'(z_0) = m\zeta_0 g'(\zeta_0)$, where m is real and $m \geq 1$.

Lemma 2. ([4]) *Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be analytic in U with $p(z) \not\equiv 1$. Let $\psi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ satisfies*
(i) *ψ is continuous in $D \subset \mathbb{C}^2$,*
(ii) $(1, 0) \in D$ *and* $\mathrm{Re}(\psi(1, 0)) > 0$,
(iii) *for all $(iu_2, v_1) \in D$ such that $v_1 \leq -(1 + u_2^2)/2$,* $\mathrm{Re}(\psi(iu_2, v_1) \leq 0$.
If $(p(z), zp'(z)) \in D$ for all $z \in D$ and $\mathrm{Re}(\psi(p(z), zp'(z)) > 0$ for all $z \in U$, then $\mathrm{Re}(p(z)) > 0\,(z \in U)$.

Let a function $L(z,t)$ be defined on $U \times \{0, \infty)$. Then $L(z,t)$ is said to be the subordination chain (or Loewner chain) if it satisfies
(i) $L(z,t)$ is analytic and univalent in U for all$t \geq 0$,
(ii) $L(z,t)$ is continuously differentiable on $t \geq 0$ for all $z \in U$,
(iii) $L(z,t_1) \prec L(z,t_2) \quad (0 \leq t_1 \leq t_2)$.

Lemma 3. ([7]) *Let* $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$ $(a_1(t) \neq 0; t \geq 0)$ *is a subordination chain if and only if it satisfies*

$$\mathrm{Re}\left\{ \frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \right\} > 0 \quad (z \in U; t \geq 0).$$

Further, we need

Lemma 4. ([5]) *Let* $\alpha \in \mathbb{C}$, $\alpha \neq 0$, *and let* $\beta \in \mathbb{C}$. *Let a function* $h(z) = c * h_1 z * h_2 z^2 + \dots$ *be analytic in* U *and* $\mathrm{Re}(\alpha h(z) + \beta) > 0\,(z \in U)$. *Then the solution of the Briot-Bouquet differential equation*

$$q(z) + \frac{zq'(z)}{\alpha q(z) + \beta} = h(z) \quad (h(0) = q(0))$$

is analytic in U *and* $\mathrm{Re}(\alpha q(z) + \beta) > 0\,(z \in U)$.

2 Subordination theorems

Applying the abouve lemmas, we derive our main theorem in

Theorem 1. *Let* $f(z) \in A$ *and* $g(z) \in A$. *Let* $g(z)$ *satisfy*
(i)$g(z)/z \neq 0\,(z \in U)$ *and* $I_{\alpha,\beta}(g(z))/z \neq 0\,(z \in U)$ *when* $\alpha \neq 1$,
(ii)$\phi(z) = (g(z)/z)^\alpha$ *satisfies*

$$\mathrm{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \quad (z \in U), \tag{2.1}$$

where $\delta < \mathrm{Re}(\alpha + \beta)$,

$$-1 < \delta \leq \frac{1 + |\alpha + \beta|^2 - \sqrt{(1 + |\alpha + \beta|^2)^2 - 4(\mathrm{Re}(\alpha + \beta))^2}}{4\mathrm{Re}(\alpha + \beta)}$$

if $\mathrm{Re}(\alpha + \beta) > 0$, *and*

$$-1 < \delta \leq \frac{1 + |\alpha + \beta|^2 + \sqrt{(1 + |\alpha + \beta|^2)^2 - 4(\mathrm{Re}(\alpha + \beta))^2}}{4\mathrm{Re}(\alpha + \beta)}$$

if $\mathrm{Re}(\alpha+\beta)<0$.
If $f(z)$ *and* $g(z)$ *satisfy the following subordination*

$$\frac{f(z)}{z} \prec \frac{g(z)}{z}, \tag{2.2}$$

then

$$\frac{I_{\alpha,\beta}(f(z))}{z} \prec \frac{I_{\alpha,\beta}(g(z))}{z}. \tag{2.3}$$

Proof. Let us define $F(z)$ and $G(z)$ by

$$F(z) = \left(\frac{I_{\alpha,\beta}(f(z))}{z}\right)^{\alpha} \quad \text{and} \quad G(z) = \left(\frac{I_{\alpha,\beta}(g(z))}{z}\right)^{\alpha},$$

respectively. Without loss of generality, we can assume that $G(z)$ is analytic and univalent in $\overline{U} = U \cup \partial U$. Otherwise, we consider, for $0<r<1$, $F(rz)/r$ and $G(rz)/r$ instead of $F(z)$ and $G(z)$, respectively.
First, we show that if the function $q(z)$ is defined by

$$q(z) = 1 + \frac{zG''(z)}{G'(z)}, \tag{2.4}$$

then $\mathrm{Re}(q(z)+\alpha+\beta)>0$ $(z\in U)$.
Since

$$I_{\alpha,\beta}(g(z)) = \left\{\frac{\alpha+\beta}{z^{\beta}}\int_0^z t^{\beta-1}g(t)^{\alpha}dt\right\}^{1/\alpha}, \tag{2.5}$$

we have

$$\alpha\frac{z(I_{\alpha,\beta}(g(z)))'}{I_{\alpha,\beta}(g(z))} = -\beta + (\alpha+\beta)\frac{\phi(z)}{G(z)}. \tag{2.6}$$

Also, we have

$$\alpha\frac{z(I_{\alpha,\beta}(g(z)))'}{I_{\alpha,\beta}(g(z))} = \alpha + \frac{zG'(z)}{G(z)}. \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$(\alpha+\beta)\phi(z) = (\alpha+\beta)G(z) + zG'(z). \tag{2.8}$$

Differentiating both sides in (2.8), we obtain

$$\beta z\phi'(z) = zG'(z)\left(\alpha + \beta + 1 + \frac{zG''(z)}{G'(z)}\right)$$

$$= zG'(z)(q(z) + \alpha + \beta).$$

This gives us that

$$1 + \frac{z\phi''(z)}{\phi'(z)} = 1 + \frac{zG''(z)}{G'(z)} + \frac{zq'(z)}{q(z)+\alpha+\beta}$$

$$= q(z) + \frac{zq'(z)}{q(z)+\alpha+\beta}.$$

If we define the function $h(z)$ by

$$h(z) = q(z) + \frac{zq'(z)}{q(z)+\alpha+\beta},$$

then, $q(0) = h(0) = 1$ and

$$\mathrm{Re}(h(z)+\alpha+\beta) = \mathrm{Re}\left(1 + \frac{z\phi''(z)}{\phi'(z)} + \alpha + \beta\right)$$

$$> -\delta + \mathrm{Re}(\alpha+\beta) > 0,$$

because δ satisfies the conditions in our theorem and $\delta \leq \mathrm{Re}(\alpha+\beta)$.
Thus applying Lemma 4, we conclude that $q(z)$ is analytic in U and $\mathrm{Re}(q(z)+\alpha+\beta) > 0$ for all $z \in U$.
Next, we show that $\mathrm{Re}(q(z)+\alpha+\beta) > 0\,(z \in U)$ implies $\mathrm{Re}q(z) > 0\,(z \in U)$.
Let us put

$$\psi(u,v) = u + \frac{v}{u+\alpha+\beta} + \delta,$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Then $\psi(u,v)$ satisfies
(i) $\psi(u,v)$ is continuous in $D = (\mathbb{C}\setminus\{-\alpha-\beta\})\times\mathbb{C}$,
(ii) $(1,0) \in D$ and $\mathrm{Re}\psi(1,0) = 1+\delta > 0$,
(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -(1+u_2^2)/2$,

$$\mathrm{Re}\psi(iu_2,v_1) = \mathrm{Re}\left(\frac{v_1}{iu_2+\alpha+\beta}\right) + \delta$$

$$= \delta - \frac{v_1\mathrm{Re}(\alpha+\beta)}{|\alpha+\beta|^2 + 2u_2\mathrm{Im}(\alpha+\beta) + u_2^2}.$$

We define

$$E_\delta(u_2) = (2\delta - \mathrm{Re}(\alpha+\beta))u_2^2 + 4\delta(\mathrm{Im}(\alpha+\beta))u_2 + 2\delta\,|\alpha+\beta|^2 - \mathrm{Re}(\alpha+\beta), \quad (2.9)$$

$$k_1 = \frac{1+|\alpha+\beta|^2 - \sqrt{(1+|\alpha+\beta|^2)^2 - 4(\mathrm{Re}(\alpha+\beta))^2}}{4\mathrm{Re}(\alpha+\beta)},$$

and

$$k_2 = \frac{1+|\alpha+\beta|^2 + \sqrt{(1+|\alpha+\beta|^2)^2 - 4(\mathrm{Re}(\alpha+\beta))^2}}{4\mathrm{Re}(\alpha+\beta)}.$$

Then, the discrimination Δ of $E_\delta(u_2)$ given by (2.8) is represented by

$$\Delta = -4(\mathrm{Re}(\alpha+\beta))^2\delta^2 + 2\delta(1+|\alpha+\beta|^2)\mathrm{Re}(\alpha+\beta) - (\mathrm{Re}(\alpha+\beta))^2.$$

Therefore, if $\mathrm{Re}(\alpha+\beta) > 0$, then $-1 < \delta \leq k_1$ implies $\Delta \leq 0$, and if $\mathrm{Re}(\alpha+\beta) < 0$, then $-1 < \delta \leq k_2$ implies $\Delta \leq 0$.
This shows that $E_\delta(u_2) \leq 0$ for all real u_2, that is, that $\mathrm{Re}\psi(iu_2, v_1) \leq 0$ for all real v_1 and u_2 such that $v_1 \leq -(1+u_2^2)/2$.
Further, we note that

$$\mathrm{Re}\psi(q(z), zq'(z)) = \mathrm{Re}\left(q(z) + \frac{zq'(z)}{q(z)+\alpha+\beta} + \delta\right)$$

$$= \mathrm{Re}\left(1 + \frac{z\phi''(z)}{\phi'(z)} + \delta\right) > 0.$$

Thus, using Lemma 2, we conclude that $\mathrm{Re}q(z) > 0$ for all $z \in YU$.
Finally, we prove that the subordination $f(z)/z \prec g(z)/z$ implies $F(z) \prec G(z)$.
Define the function $L(z,t)$ by

$$L(z,t) = G(z) + \frac{1+t}{\alpha+\beta}zG'(z) \quad (t \geq 0). \tag{2.10}$$

Note that $G'(0) = 1$ and

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(1 + \frac{1+t}{\alpha+\beta}\right) = 1 + \frac{1+t}{\alpha+\beta} \neq 0.$$

This shows that the function

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$$

satisfies $a_1(t) \neq 0$ for all $t \geq 0$.
Furthermore, we have

$$\mathrm{Re}\left\{\frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} = \mathrm{Re}\left(\alpha+\beta+(1+t)\left(1+\frac{zG''(z)}{G'(z)}\right)\right\}$$

$$= \mathrm{Re}(q(z)+\alpha+\beta) + t\mathrm{Re}q(z) > 0$$

for all $z \in U$. Therefore, by virtue of Lemma 3, $L(z,t)$ is the subordination chain. Note that

$$\phi(z) = G(z) + \frac{1}{\alpha+\beta} zG'(z) = L(z,0)$$

and

$$L(z,0) \prec L(z,t) \quad (t \geq 0)$$

from the definition of the subordination chain. Next, we support that $F(z)$ is not subordinate to $G(z)$. Then, there exists points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that

$$F(|z| < |z_0|) \subset G(U) \quad \text{and} \quad F(z_0) = G(\zeta_0).$$

This implies that $L(\zeta 0, t) \notin L(U, t)$.
Since, by Lemma 1,

$$z_0 F'(z_0) = (1+t)\zeta_0 G'(\zeta_0) \quad (t \geq 0),$$

we have

$$\begin{aligned} L(\zeta_0, t) &= G(\zeta_0) + \frac{1+t}{\alpha+\beta}\zeta_0 G'(\zeta_0) \\ &= F(z_0) + \frac{1}{\alpha+\beta} z_0 F'(z_0) \\ &= \left(\frac{f(z_0)}{z_0}\right)^{\alpha} \in \psi(U), \end{aligned}$$

because $f(z)/z \prec g(z)/z$. This contradicts that $L(\zeta_0, t) \notin L(U, t)$. Therefore, the subordination $f(z)/z \prec g(z)/z$ has to imply $F(z) \prec G(z)$.
Now, since

$$I_{\alpha,\beta}(g(z)) = 1 + c_1 z + c_2 z^2 + \ldots \neq 0 \quad (z \in U)$$

when $\alpha \neq 1$, we conclude that

$$F(z) = \left(\frac{I_{\alpha,\beta}(f(z))}{z}\right)^{\alpha} \prec G(z) = \left(\frac{I_{\alpha,\beta}(g(z))}{z}\right)^{\alpha}$$

gives that

$$\frac{I_{\alpha,\beta}(f(z))}{z} \prec \frac{I_{\alpha,\beta}(g(z))}{z}.$$

This completes the proof of our theorem.

□

If we take α and β such that $\alpha + \beta = 1$ in Theorem 1, then $-1 < \delta \leq 1/2$. Therefore, we have

Corollary 1. *Let $f(z) \in A$ and $g(z) \in A$. Let $g(z)$ satisfy*
(i)$g(z)/z \neq 0$ $(z \in U)$ *and* $I_{\alpha,1-\alpha}(g(z))/z \neq 0$ $(z \in U)$ *when* $\alpha \neq 1$,
(ii)$\phi(z) = (g(z)/z)^{\alpha}$ *satisfies*

$$\mathrm{Re}\left(1 + \frac{z\phi''(z)}{\phi'(z)}\right) > -\frac{1}{2}.$$

Then

$$\frac{f(z)}{z} \prec \frac{g(z)}{z} \Longrightarrow \frac{I_{\alpha,1-\alpha}(f(z))}{z} \prec \frac{I_{\alpha,1-\alpha}(g(z))}{z},$$

where

$$I_{\alpha,1-\alpha}f(z) = \left\{\frac{1}{z^{1-\alpha}}\int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt\right\}^{1/\alpha}.$$

If we make α and β such that $\alpha + \beta = -1$ in Theorem 1, then$-1 < \delta \leq -1/2$. Thus we have

Corollary 2. *Let $f(z) \in A$ and $g(z) \in A$. Let $g(z)$ satisfy*
(i)$g(z)/z \neq 0$ $(z \in U)$ *and* $I_{\alpha,-1-\alpha}(g(z))/z \neq 0$ $(z \in U)$ *when* $\alpha \neq 1$,
(ii)$\phi(z) = (g(z)/z)^{\alpha}$ *satisfies*

$$\mathrm{Re}\left(1 + \frac{z\phi''(z)}{\phi'(z)}\right) > \frac{1}{2}.$$

Then

$$\frac{f(z)}{z} \prec \frac{g(z)}{z} \Longrightarrow \frac{I_{\alpha,-1-\alpha}(f(z))}{z} \prec \frac{I_{\alpha,-1-\alpha}(g(z))}{z},$$

where

$$I_{\alpha,-1-\alpha}(f(z)) = \left\{-z^{1+\alpha}\int_0^z \frac{1}{t^2}\left(\frac{f(t)}{t}\right)^{\alpha} dt\right\}^{1/\alpha}.$$

Taking α and β such that $\alpha + \beta = 1 + i$ in Theorem 1, we have

Corollary 3. *Let $f(z) \in A$ and $g(z) \in A$. Let $g(z)$ satisfy*
(i)$g(z)/z \neq 0$ $(z \in U)$ *and* $I_{\alpha,1+i-\alpha}(g(z))/z \neq 0$ $(z \in U)$ *when* $\alpha \neq 1$,
(ii)$\phi(z) = (g(z)/z)^{\alpha}$ *satisfies*

$$\mathrm{Re}\left(1 + \frac{z\phi''(z)}{\phi'(z)}\right) > \frac{\sqrt{5}-3}{4} \fallingdotseq -0.190983.$$

Then

$$\frac{f(z)}{z} \prec \frac{g(z)}{z} \Longrightarrow \frac{I_{\alpha,1+i-\alpha}(f(z))}{z} \prec \frac{I_{\alpha,1+i-\alpha}(g(z))}{z},$$

where

$$I_{\alpha,1+i-\alpha}(f(z)) = \left\{\frac{1+i}{z^{1+i-\alpha}}\int_0^z t^i \left(\frac{f(t)}{t}\right)^{\alpha} dt\right\}^{1/\alpha}.$$

Further, letting $\alpha+\beta=-1+i/2$ in Theorem 1, we have

Corollary 4. *Let $f(z)\in A$ and $g(z)\in A$. Let $g(z)$ satisfy*
(i)$g(z)/z\neq 0$ $(z\in U)$ and $I_{\alpha,-1+i/2-\alpha}(g(z))/z\neq 0$ $(z\in U)$ when $\alpha\neq 1$,
(ii)$\phi(z)=(g(z)/z)^{\alpha}$ satisfies

$$\mathrm{Re}\left(1+\frac{z\phi''(z)}{\phi'(z)}\right)>-\frac{9+\sqrt{17}}{16}\fallingdotseq -0.8201941.$$

Then

$$\frac{f(z)}{z}\prec\frac{g(z)}{z}\Longrightarrow\frac{I_{\alpha,-1+i/2-\alpha}(f(z))}{z}\prec\frac{I_{\alpha,-1+i/2-\alpha}(g(z))}{z},$$

where

$$I_{\alpha,-1+i/2-\alpha}(f(z))=\left\{\frac{-2+i}{2z^{-1+i/2-\alpha}}\int_0^z t^{-1+i/2}\left(\frac{f(t)}{t}\right)^{\alpha}dt\right\}^{1/\alpha}.$$

3 An application of hypergeometric functions

For complex numbers a,b, and c with $c\neq 0,-1,-2,\ldots$, the hypergeometric functin ${}_2F_1(a,b;c;z)$ is defined by

$${}_2F_1F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined, in terms of gamma functions, by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}\lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1) & (n\in\mathbb{N}=\{1,2,3,\ldots\})\\ 1 & (n=0).\end{cases}$$

It is well-known that

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}dt,$$

for $\mathrm{Re}(c)>\mathrm{Re}(a)>0$.
If we consider

$$g(z)=\frac{z}{(1-z)^k}\qquad(k\in\mathbb{C}),$$

then we have

$$I_{\alpha,\beta}(g(z))=\left\{\frac{\alpha+\beta}{z^{\beta}}\int_0^z t^{\alpha+\beta-1}(1-t)^{-\alpha k}dt\right\}^{1/\alpha}$$

$$= \left\{ (\alpha+\beta) z^{\alpha} \int_0^1 u^{\alpha+\beta-1}(1-zu)^{-\alpha k} du \right\}^{1/\alpha}$$
$$= z_2F_1(\alpha+\beta, \alpha k; \alpha+\beta+1; z)^{1/\alpha}$$

with $\mathrm{Re}(\alpha+\beta) > 0$. Moreover, we note that

$$\frac{g(z)}{z} = \frac{1}{(1-z)^k} \neq 0 \qquad (z \in U).$$

Applying the above to Theorem 1, we obtain

Theorem 2. *Let $f(z) \in U$. Let $\alpha > 0$ and β be a complex number such that* $\mathrm{Re}(\alpha+\beta) > 0$. *If $f(z)$ satisfies the subordination*

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^k},$$

then

$$\frac{I_{\alpha,\beta}(f(z))}{z} \prec {}_2F_1(\alpha+\beta, \alpha k; \alpha+\beta+1; z)^{1/|alpha},$$

where

$$1 - \frac{1+|\alpha+\beta|^2 - \sqrt{(1+|\alpha+\beta|^2)^2 - 4(\mathrm{Re}(\alpha+\beta))^2}}{2\mathrm{Re}(\alpha+\beta)} \leq \alpha k < 3.$$

Taking $\alpha > 0$ and $\beta = 1-\alpha$ in Theorem 2, we have

Example 1. *For $f(z) \in A$ and $0 \leq \alpha k < 3$,*

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^k} \Longrightarrow \frac{I_{\alpha,1-\alpha}(f(z))}{z} \prec {}_2F_1(1, \alpha k; 2; z)^{1/\alpha},$$

where

$$I_{\alpha,1-\alpha}(f(z)) = \left\{ \frac{1}{z^{1-\alpha}} \int_0^z \left(\frac{f(t)}{t} \right)^{\alpha} dt \right\}^{1/\alpha}.$$

Finally, if we make $\alpha > 0$ and $\beta = 1+i-\alpha$ in Theorem 2, then we have

Example 2. *For $f(z) \in A$ and $(\sqrt{5}-1)/2 \leq \alpha k < 3$,*

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^k} \Longrightarrow \frac{I_{\alpha,1+i-\alpha}(f(z))}{z} \prec {}_2F_1(1+i, \alpha k; 2+i; z)^{1/\alpha},$$

where

$$I_{\alpha,1+i-\alpha}(f(z)) = \left\{ \frac{1+i}{z^{1+i-\alpha}} \int_0^z t^i \left(\frac{f(t)}{t} \right)^{\alpha} dt \right\}^{1/\alpha}.$$

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